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LETTER TO THE EDITOR

Critical behaviour of the kinetic Ising model on a fractal lattice

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Abstract. We study the critical behaviour (static and dynamic) of the Ising model, defined on a fractal Sierpinski carpet. Using Monte Carlo calculations estimates for various critical exponents are obtained from a finite-size scaling analysis. Our results are compared with the available predictions for the dynamical critical exponent z. We find the dynamic exponent z near 2.2 to be lower than predicted.

Dynamic phenomena associated with fractal geometries have been the object of various studies in recent years (anomalous diffusion, transport in random media, and so on). On the other hand, a systematic study of static critical phenomena on fractal lattices has been carried out. More recently, the critical dynamics of the kinetic Ising model on fractal lattices (Achiam 1985a, b, Luscombe and Desai 1985, Rammal 1985) and percolation clusters (Henley 1985, Rammal and Benoît 1985a, b) has been investigated. Some of the results obtained have received numerical confirmation (Rammal and Benoît 1985b, Chowdhury and Stauffer 1985, Jain 1985) particularly for the percolation clusters, which are known to be relevant for diluted magnetic systems (Stinchcombe 1983). Furthermore, a lower bound for the dynamic critical exponent z on finitely ramified fractals has been suggested (Luscombe and Desai 1985): $z \geqslant \bar{d}$, where $\bar{d}$ refers to the fractal dimension of the lattice. The value $z = \bar{d} + (1/\nu)$ has been obtained for quasilinear fractals (Achiam 1985a,b). Finally, for infinitely ramified fractals such as the Sierpinski carpet, a domain wall argument has been used (Rammal 1985) to obtain $z = 1 + \bar{d}$, which is argued to hold at $\bar{d} = 1 + \varepsilon$, where $T_c \sim \varepsilon$, $\nu \sim 1/\varepsilon$ and $\varepsilon \ll 1$.

Our main purpose in this letter is to present the results of a numerical study of the critical behaviour (static and dynamic) of the Ising model on a fractal lattice. We used a fractal structure of the Sierpinski carpet type, with parameters $b = 1$, $c = 3$ and $l = 1$. The static critical behaviour of this type of fractal lattice has been studied previously both numerically (Bhanot *et al* 1984) and using the Migdal-Kadanoff decimation scheme (Gefen *et al* 1983). The fractal dimension of this lattice is given by $\bar{d} = \ln[b^2(c^2 - l^2)]/\ln(bc)$. In the case considered here, $\bar{d} = \ln 8/\ln 3 \simeq 1.892\,78$. The carpet has been chosen for two reasons: firstly because of the finite value of the critical temperature T_c and secondly because of the presence of holes inside the lattice, and this occurring at all length scales. At stage $n \geqslant 1$ of iteration, the linear length scale of the embedding lattice is $L = 1 + 3^n$ and the total number of sites is $N_n = (44 \times 8^n + 56 \times 3^n + 40)/35$. For large n, the ratio of the number of sites having a coordination number 3 and 4 has a limiting finite value given by $\frac{35}{352} \simeq 0.099\,43$.

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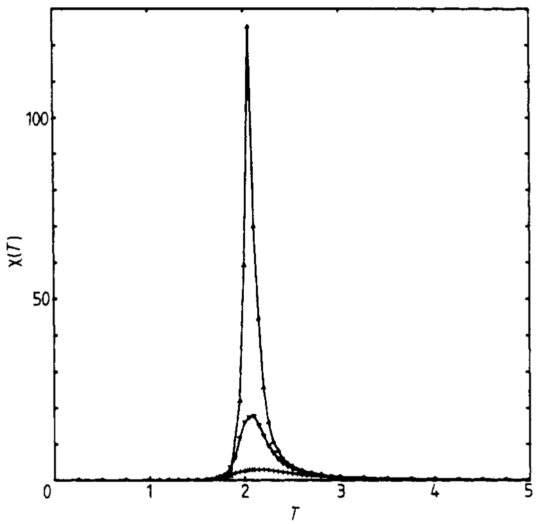

Figure 1. Magnetic susceptibility per spin $\chi(T)$ as a function of temperature T, for three lattice sizes $N = 98$ ($n = 2$), $N = 688$ ($n = 3$) and $N = 5280$ ($n = 3$), corresponding to stage iteration n; +: $N = 96$; $\nabla = N = 688$; $\triangle$: $N = 5280$.

On the sites of the Sierpinski carpet are located Ising spins, coupled by a nearest-neighbour interaction J: $\mathscr{H} = -J\,\Sigma_{\langle ij\rangle}\,\sigma_i\sigma_j$, $\sigma_i = \pm 1$. The same coupling constant J has been used for internal and boundary bonds. Critical temperature and critical exponents have been obtained, through a finite-size scaling, over three-stage iterations. The corresponding lattices have respectively $N = 96$ ($n = 2$), $N = 668$ ($n = 3$) and $N = 5280$ ($n = 4$) sites. The results presented below have been extracted from a Monte Carlo simulation, performed up to 10^5 MCS/spin. In figures 1 and 2 are shown the results for the magnetic susceptibility $\chi(T)$ and for the specific heat $C(T)$ corresponding to

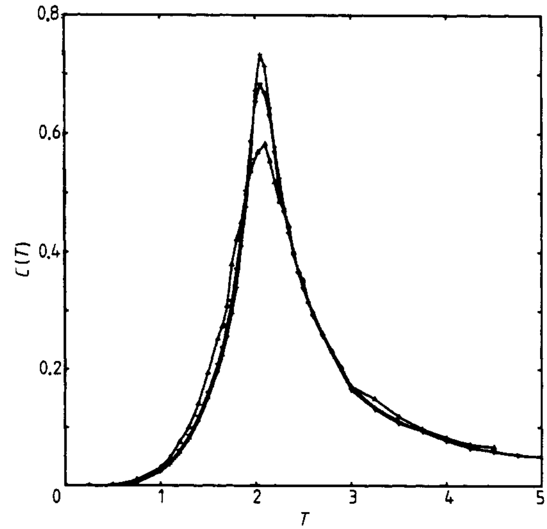

Figure 2. Same plot as in figure 1, for the specific heat $C(T)$. We have used the same symbols for the different sizes.

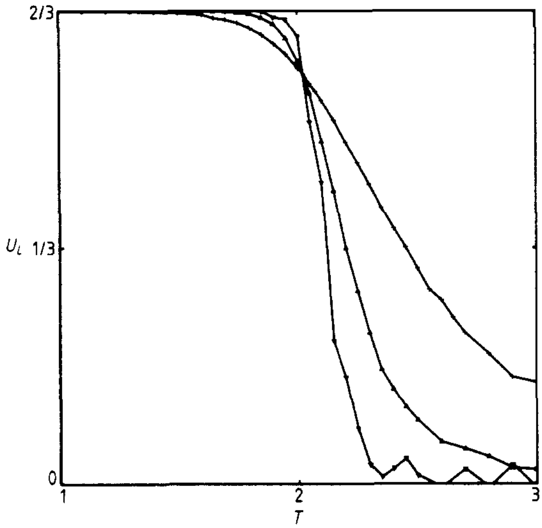

Figure 3. Temperature variation of the renormalised coupling constant U_L.

the three sizes studied. For both $\chi(T)$ and $C(T)$ a sharp peak occurs at $T \sim 2J$. An accurate value for the critical temperature T_c has been determined, using the renormalised coupling constant method (Binder 1985). For this we have calculated, during the Monte Carlo simulations, the first moments of the thermal equilibrium values of the normalised magnetisation $m = N^{-1} \Sigma_i \sigma_i$. More precisely, $\langle|m|\rangle$, $\langle m^2\rangle$ and $\langle m^4\rangle$ have been recorded and the renormalised coupling constant U_L has been calculated:

$$U_L = 1 - \langle m^4\rangle_L / 3\langle m^2\rangle_L^2.$$

The behaviour of this reduced cumulant makes it rather suitable for obtaining estimates for T_c which are not biased by any assumption about the critical exponents. In figure 3 are shown the results thus obtained for the temperature variation of U_L, for the three-stage iteration ($n = 2,3$ and 4). The common intersection point of these three curves yields the following value for T_c: $T_c/J = 2.02 \pm 0.02$ ($J/T_c = 0.495$). This value is to be compared with the estimates $J/T_c = 0.511 \pm 0.03$, 0.500 ± 0.005 and 0.498 ± 0.05 obtained by Bhanot *et al* (1984) for the carpet $b = 2$, $c = 4$ and $l = 2$ ($\bar{d} = \ln 48/\ln 8 = 1.8616$). Note that both values of T_c are lower than the critical temperature of the 2D Ising model: $J/T_c = 0.44$. The numerical results agree therefore with the increase of T_c with $\bar{d}$, from $T_c = 0$ at $\bar{d} = 1$ (linear Ising chain) to the 2D limiting value.

The correlation length exponent ν, associated with the transition at $T = T_c$, can be deduced from the behaviour of U_L close to the fixed point U^* $(= 0.577)$:

$$1/\nu = \left.\frac{\ln(\partial U_{bL}/\partial U_L)}{\ln b}\right|_{U^*}$$

For the lattices studied here, it is more convenient to use the following equivalent form:

$$1/\nu\bar{d} = \left.\frac{\ln[\partial U_{n+1}/\partial U_n]}{\ln[N_{n+1}/N_n]}\right|_{U^*}$$

giving directly the intrinsic product $\nu\bar{d}$. Here N_n denotes the number of spins at stage n. The extracted values of ν and $\nu\bar{d}$ are respectively $\nu = 1.0904$ and $\nu\bar{d} = 2.063$. Using the hyperscaling relation $2-\alpha=\bar{d}\nu$, one deduces the following negative value for the specific heat exponent: $\alpha=-0.06$.

It is interesting to notice that the ε-expansion value of ν, at $\bar{d}=1+\varepsilon$, gives $\nu=1/(\bar{d}-1)$ and then $\nu=1.1202$ at $\bar{d}=1.892\,78$ and $\nu=1.1606$ at $\bar{d}=1.8616$ respectively. This value of ν in our case ($\bar{d}=1.892\,78$) is slightly higher than the numerical one. This contrasts with the findings of Bhanot *et al* where $\nu=1.28$ has been obtained from the decay of the spin-spin correlation function along lines which have encountered no holes. We believe that the value of ν thus obtained here is the appropriate one and actually refers to the averaged correlation function over all pairs of spins at a given distance. This distinction would arise in any inhomogeneous structure, such as the carpet considered here.

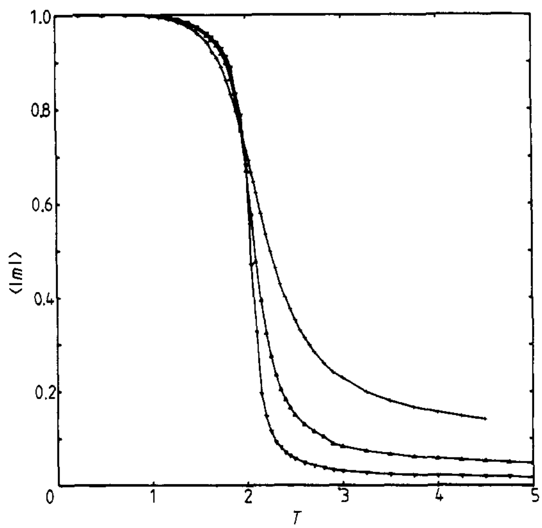

Figure 4. Magnetisation per spin as a function of the temperature T.

In figure 4 the temperature variations of the magnetisation $m(T)$ are shown for the three lattice sizes N. The same data are used in figure 5 in a finite-size scaling plot, $mN^{\beta/\nu\bar{d}}$ as a function of $(T-T_c)N^{1/\nu\bar{d}}$, in order to extract the ratio $\beta/\nu\bar{d}$. The best fit in the plot leads to $\beta/\nu\bar{d}=0.045$ and then a rather small value for the exponent β: $\beta=0.0928$.

The same procedure has been used for the magnetic susceptibility $\chi(T)$, as shown in figure 6, where $\chi N^{-\gamma/\nu\bar{d}}$ is given as a function of $(T-T_c)N^{1/\nu\bar{d}}$. The best fit in this plot gives $\gamma/\nu\bar{d}=0.925$ and then $\gamma=1.908$. The accuracy of our estimate for different exponents (β, γ, ν) is easily checked on the value of $(2\beta+\gamma)/\bar{d}\nu=1.015$, instead of 1.000 according to scaling relations. The set of exponents we have obtained in this letter is summarised in table 1, where the results of Bhanot *et al* have also been included. As can be seen a monotonic variation of the exponents is observed, when $\bar{d}$ increases from $\bar{d}=1$ to $\bar{d}=2$. Note that η and α given here were extracted from the scaling relations $2-\alpha=\bar{d}\nu$ and $2-\eta=\gamma/\nu$ respectively.

Using the single-spin flip dynamics, we have studied the critical relaxation on the Sierpinski carpet. For this we have calculated the non-linear relaxation times of the magnetisation $m(t)$ and the energy $E(t)$. Our simulation always starts with all spins

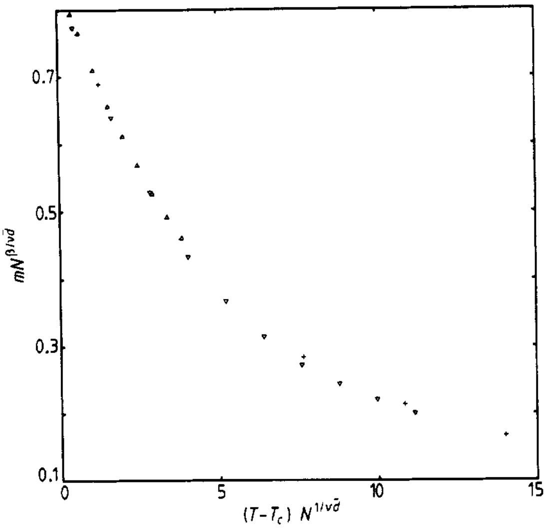

Figure 5. Finite-size scaling plot for the magnetisation $m(T)$, at different lattice sizes.

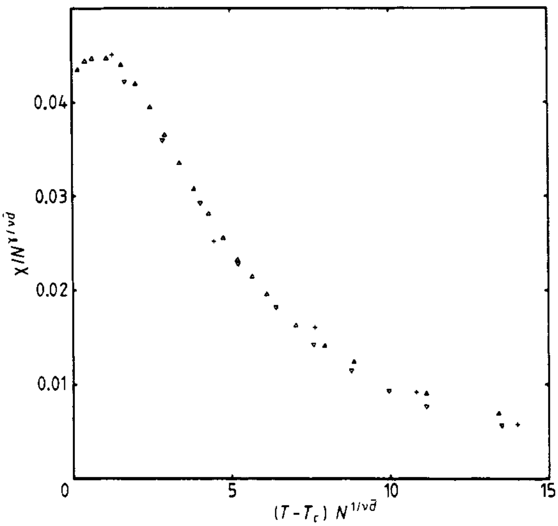

Figure 6. The same plot as in figure 5 for the magnetic susceptibility.

Table 1. Numerical values of the critical temperature T_c and static critical exponents for two Sierpinski carpets of fractal dimension $\bar{d}$. For comparison, the corresponding value for 1D Ising model ($\bar{d}=1$) and 2D Ising model ($\bar{d}=2$) are included.

$\bar{d}$	T_c	ν	β	γ	η	α	z
1	0	(1)	(0)	(1)	(1)	(1)	2
$1+\varepsilon$	ε	$1/\varepsilon$	—	—	—	—	$2+\varepsilon$
1.8616	1.957 0.005	1.28 0.05	0.10 0.05	1.90 0.05	0.5156	−0.10	—
1.8927	2.03 0.02	1.090	0.0928	1.908	0.3155	−0.063	2.20 ± 0.06
2	2.269	1	$\frac{1}{8}$	$\frac{7}{4}$	$\frac{1}{4}$	O(ln)	2.12 ± 0.06

up, $m(0)=1$. The non-linear relaxation time for the magnetisation $m(t)$ is defined as usual by

$$\tau_M^{nl\,\prime} = \int_0^\infty \mathrm{d}t \frac{m(t)-m(\infty)}{1-m(\infty)}.$$

Simililarly, for the energy relaxation, we have used

$$\tau_E^{nl} = \int_0^\infty \mathrm{d}t \frac{E(t)-E(\infty)}{E(0)-E(\infty)}.$$

In order to achieve sufficient accuracy, the present calculations were restricted to $T=T_c$. The dynamic exponents have been extracted using finite-size scaling. Indeed, scaling arguments show that at $T \gtrsim T_c$, τ_M and τ_E increase according to $\tau_M \sim \xi^z$ and $\tau_E \sim \xi^z$ respectively. The non-linear exponents are related to the linear exponent z (Racz 1976) through $z_M = z - \beta/\nu$ and $z_E = z-(1-\alpha)/\nu$ respectively. Here z is assumed to be the same for the magnetisation and for the energy relaxation. At critical temperature and for a finite system, this leads to the following expressions for τ_M and τ_E as functions of the size N:

$$\tau_M \sim N^{x_M} \qquad x_M = (z-\beta/\nu)/\bar{d}$$

$$\tau_E \sim N^{x_E} \qquad x_E = \left(z - \frac{1-\alpha}{\nu}\right)(\bar{d})^{-1}.$$

Only the exponents x_M and x_E are measured here. Let us first consider the relaxation of $m(t)$. τ_M, as defined above, has been measured for three-stage iterations. From the log-log plot of τ_M averaged over 3000 relaxations as a function of N (figure 7),

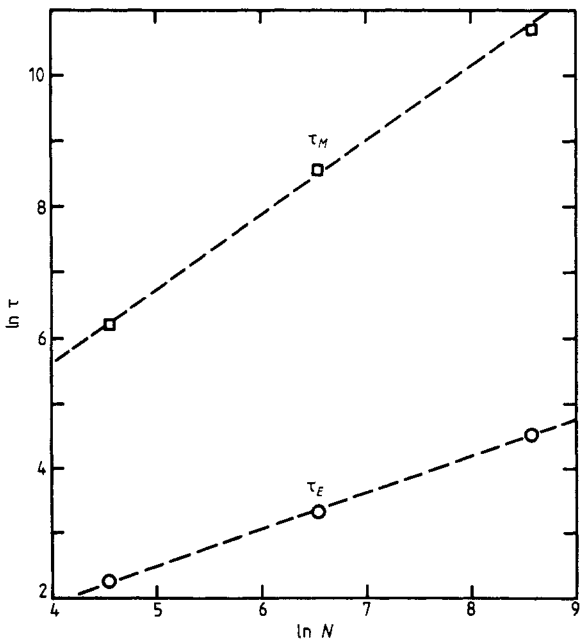

Figure 7. Log-log plot of the relaxation times (τ_M and τ_E) as functions of the lattice size N.

we have deduced the value for the exponent to be $x_M = 1.113$. Using the previously obtained values for β and ν, one obtains $z = 2.20 \pm 0.06$.

The same analysis has been carried out for the energy relaxation. We obtained $x_E = 0.573$, which leads to $z = 2.06 \pm 0.06$.

The value of z (at least that deduced from z_M) is larger than the most accurate one: $z = 2.12 \pm 0.06$ (Jan *et al* 1983, Kalle 1984) for the 2D Ising model. The increase of the dynamic exponent is in good agreement with the recent predictions summarised at the beginning of this letter. However the values for z obtained here are clearly lower than the predicted ones. Indeed, the first one, $z = \bar{d} + 1/\nu$, leads to $z = 2.8098$ whereas the second one gives $z = \bar{d} + 1 = 2.8927$.

Let us conclude this letter with two remarks. Firstly, the static critical exponents on the Sierpinski carpet appear to follow the expected pattern and this is shown in table 1. Similarly, due to the presence of holes in the lattice (defective structure), the dynamic critical exponent z is increased, in comparison with the 2D Ising model. This means that details of the structure not described by $\bar{d}$ alone must be involved in the calculation of z. Secondly, the obtained value of z does not fit the recently made predictions for its value, and this calls for further investigation.

References

Achiam Y 1985a *Phys. Rev.* B **31** 4732

—— 1985b *Phys. Rev.* B **32** 1796

Bhanot G, Neuberger H and Shapiro J A 1984 *Phys. Rev. Lett.* **53** 2277

Binder K 1985 *J. Comput. Phys.* **59** 1

Chowdhury D and Stauffer D 1985 *J. Phys. A: Math. Gen.* **19** L19

Gefen Y, Meir Y, Mandelbrot B B and Aharony A 1983 *Phys. Rev. Lett.* **50** 145

Henley C 1985 *Phys. Rev. Lett.* **54** 2030

Jain S 1985 *J. Phys. A: Math. Gen.* **19** L57

Jan N, Moseley L L and Stauffer D 1983 *J. Stat. Phys.* **33** 1

Kalle C 1984 *J. Phys. A: Math. Gen.* **17** L801

Luscombe J H and Desai R C 1985 *Phys. Rev.* B **32** 488

Racz Z 1976 *Phys. Rev.* B **13** 163

Rammal R 1985 *J. Physique* **46** 1837

Rammal R and Benoit A 1985a *Phys. Rev. Lett.* **55** 649

—— 1985b *J. Physique Lett.* **46** L667

Stinchcombe R B 1983 *Phase Transitions and Critical Phenomena* vol 7, ed C Domb and J L Lebowitz (New York: Academic)